Eigenfunctions of spinless particles in a one-dimensional linear potential well

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Abstract

In the present paper, we work out the eigenfunctions of spinless particles bound in a one-dimensional linear finite range, attractive potential well, treating it as a time-like component of a four-vector. We show that the one-dimensional stationary Klein-Gordon equation is reduced to a standard differential equation, whose solutions, consistent with the boundary conditions, are the parabolic cylinder functions, which further reduce to the well-known confluent hypergeometric functions.

Keywords: Linear Potential, Klein-Gordon Equation, Eigen Functions, Parabolic Cylinder Functions, Confluent Hypergeometric Functions.

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1 Introduction

With ever-increasing applicability of relativistic wave equations in nuclear physics and other areas, the relativistic bound state solutions of the Klein-Gordon and Dirac equations for various potentials has drawn the attention of researchers. Setting apart the mathematical complexity and computational difficulty, there are certain unresolved questions in relativistic theory for treating a general potential. In fact, the way of incorporating a general potential is not unambigiously defined.

In literature, several authors [1 - 5] have addressed the bound states of various kinds of linear potential. While Chiu [6] has examined the quarkonium systems with the regulated linear plus Coulomb potential in momentum space, Deloff [7] has used a semi-spectral Chebyshev method for numerically solving integral equations and has applied the same to the quarkonium bound state problem in momentum space.

In recent years, Rao and Kagali [8 - 10] have analysed the bound states of spin-half and spin-zero particles in a screened Coulomb potential, having a linear behaviour near the origin and shown the existence of genuine bound states. Very recently, Rao and Kagali[11] have reported on the bound states of a non-relativistic particle in a finite, short-range linearly rising potential, envisaged as a quark-confining potential. In the present paper, we explore the relativistic bound states of spinless particles in the one-dimensional linear potential by considering the celebrated Klein-Gordon equation.

2 The Klein-Gordon Equation with the Linear Potential

Since the early days of quantum mechanics, the relativistic investigation of various one-dimensional systems is considered to be important. The Klein-Gordon equation which essentially describes spin zero particles like the pions and kaons, is a second order wave equation in space and time and indeed a Lorentz invariant. Presently, we explore the solutions of the stationary Klein-Gordon equation with the linear potential well, treating it as a time-like component of a four-vector.

The one-dimensional time-independent form of the Klein-Gordon equation for a free particle of mass $'m'$, is

$$\left[\frac{d^2}{dx^2} + \frac{E^2 - m^2c^4}{c^2\hbar^2}\right]\psi(x) = 0 \tag{1}$$

For a general potential $V(x)$, treated as the fourth component of a Lorentz-vector, this equation takes the form[12]

$$\left[\frac{d^2}{dx^2} + \frac{\left(E - V(x)\right)^2 - m^2c^4}{c^2\hbar^2}\right]\psi(x) = 0. \tag{2}$$

Thus for the potential $V(x)$, in the vector-coupling scheme, the above equation may be written as

$$\left[\frac{d^2}{dx^2}+\frac{E^2-2EV(x)+V^2(x)-m^2c^4}{c^2\hbar^2}\right]\psi=0. \tag{3}$$

Interestingly, the above equation may be rewritten in the Schrodinger form, with an effective energy and effective potential as

$$\left[\frac{d^2}{dx^2}+(E_{eff}-V_{eff})\right]\psi=0 \tag{4}$$

with $E_{eff}=\frac{E^2-m^2c^4}{c^2\hbar^2}$ and $V_{eff}=\frac{2EV(x)-V^2(x)}{c^2\hbar^2}$.

Since E_{eff} and V_{eff} are non-linear in E and V, some novel results may be expected. As in the non-relativistic case, the allowed free-particle solution, outside the potential boundry, would yield

$$\psi_1(x)=C_1e^{\alpha x} \qquad -\infty<x\le -a \tag{5}$$

$$\psi_4(x)=D_1e^{-\alpha x} \qquad a\le x<\infty, \tag{6}$$

consistent with the requirement $\psi(x)$ vanishes as $|x|\to\infty$. Here α^2 =- E_{eff} is implied.

To discuss the nature of the solution within the potential region, $-a<x<a$, we consider a simple linear rising, finite range potential of the form [11]

$$V(x)=-\frac{V_0}{a}(a-|x|) \tag{7}$$

in which the well depth V_0 and range $2a$ are positive and adjustable parameters. Owing to its shape, this potential could also be called the trianglular potential well. The linear, finite-ranged potential so constructed, serves as a good model to describe the energy specrum of particles, both relativistically and non-relativistically. We have recently reported that this potential has a rich set of solutions and can bind non-relativistic particles. Presently, we study the bound states of spin zero particles in this linear potential, treating it as a Lorentz vector.

Introducing the potential in Eqn.(3) and on simplification, we obtain, for $x>0$,

$$\left[\frac{d^2}{dx^2}+\frac{1}{c^2\hbar^2}\left\{\frac{V_0^2}{a^2}x^2-\left(2EV_0+2V_0^2\right)\frac{x}{a}+(E+V_0)^2-m^2c^4\right\}\right]\psi=0 \tag{8}$$

This equation may be written as

$$\left[\frac{d^2}{dx^2}+\frac{A}{a^2}\left(\frac{x^2}{a^2}\right)+\frac{B}{a^2}\left(\frac{x}{a}\right)+\frac{C}{a^2}\right]\psi=0 \tag{9}$$

where $A=\bar{V}_0^2$, $B=-2\bar{E}\bar{V}_0-2\bar{V}_0^2$ and $C=\left(\bar{E}+\bar{V}_0\right)^2-\bar{m}^2$.

Here $\bar{V}_0 = \frac{V_0}{\hbar c/a}$, $\bar{E} = \frac{E}{(\hbar c/a)}$ and $\bar{m} = \frac{mc^2}{\hbar c/a}$.

It is trivial to note that $\bar{V}_0$, $\bar{E}$ and $\bar{m}$ are all dimensionless quantities.

Defining a new variable

$$y = \frac{x}{a},$$

Eqn.(9) transforms into a standard form [13]

$$\frac{d^2\psi}{dy^2} + \left(Ay^2 + By + C\right)\psi = 0, \tag{10}$$

whose solutions are the well-known Parabolic Cylinder Functions.

Further, with the substitution $z = 2\sqrt{A}\left(y + \frac{B}{2A}\right)$, the above equation takes the form

$$4A\frac{d^2\psi}{dz^2} + \left(\frac{z^2}{4} - D\right)\psi = 0 \tag{11}$$

where $D = \frac{B^2 - 4AC}{4A}$ is implied.

Using the transformation $\rho^2 = \frac{z^2}{\sqrt{4A}}$, we obtain

$$\frac{d^2\psi}{d\rho^2} + \left(\frac{\rho^2}{4} - b\right)\psi = 0 \tag{12}$$

with $b = \frac{B^2 - 4AC}{(4A)^{\frac{3}{2}}}$.

It is straightforward to check that $b = \frac{\bar{m}^2}{2\bar{V}_0} > 0$, since both $\bar{m}$ and $\bar{V}_0$ are positive. The eigenfunctions of spinless particles, which are the solutions of Eqn.(12) may be written in terms of the confluent hypergeometric functions as

$$\psi = N\ exp\left(-\frac{1}{4}\rho^2 e^{\frac{i\pi}{2}}\right) M\left(\frac{-ib}{2} + \frac{1}{4}, \frac{1}{2}, \frac{1}{2}\rho^2 e^{\frac{i\pi}{2}}\right) \tag{13}$$

Physically admissible solutions require finiteness and normalizability and as is evident from the above equation, we see that the wavefunction vanishes as $|x| \to \infty$, and thus being square integrable, represents genuine bound states.

3 Results and Discussion

In relativistic quantum mechanics, it is well-known that a general potential can be introduced in the wave equation in two different ways following the *minimal coupling scheme*. While in vector coupling, the potential $V(x)$ is treated as the fourth component of a four vector field, in scalar coupling, it is added to the invariant mass. Whereas the vector interaction is charge dependent and acts differently on particles and antiparticles, the scalar interaction is independent of the charge of the particle and has the same

effect on both particles and antiparticles. Hence, it is interesting to study the quantum dynamics of relativistic particles for various interactions using different coupling schemes, with a view to decide on the appropriate prescription for a given potential.

The linear potential, envisaged as a quark-confining potential, is central in particle physics. Our investigation concerning the boundstates of spinless particle in the one-dimensional linear, finite-range potential, is seemingly interesting. It is trivial to note that the Klein Gordon equation can be reduced to a Schrodinger-like equation with an effective energy E_{eff} and an effective potential V_{eff}. The illuminating relation between the Klein Gordon equation and the Schrodinger equation with an equivalent energy dependent potential has a number of applications. If the potential is weak enough to ignore the V^2 term, the relativistic formalism becomes equivalent to the non-relativistic formalism. More importantly, in situations where the Klein-Gordon equation is not exactly solvable, the Schrodinger form of the KG equation sheds some light on the problem as it could be reduced to a solvable eigenvalue problem.

In the present work, we show that the one-dimensional Klein Gordon equation for the linear potential in the vector-coupling scheme is reduced to a standard differential equation, whose solutions, consistent with the boundary condition are the parabolic cylinder functions, which on further simplification yield the confluent hypergeometric functions. Apart from being elegant, the vector coupling prescription is particularly significant in the sense that it preserves gauge invariance. Such studies, apart from being pedagogical in nature, are potentially exciting and significant.

The linear potential well so described, has potential applications in electronics[14]. It would be interesting to study the Dirac bound states of such a linearly rising potential of finite range, which would serve as a good model to describe the quarkonia.

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